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On the integrability of stationary and restricted flows of the KdV hierarchy*

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Abstract. A bi-Hamiltonian formulation for stationary flows of the KdV hierarchy is derived in an extended phase space. A map between stationary flows and restricted flows is constructed: in one case it connects an integrable Hénon–Heiles system and the Garnier system. Moreover, a new integrability scheme for Hamiltonian systems is proposed that holds in the standard phase space.

1. Introduction

In recent years there has been an increasing interest for the construction of finite-dimensional dynamical systems from soliton equations, through the so-called methods of *stationary flows* and *restricted flows* (see [1, 2] and references therein). The discovery of suitable sets of coordinates has allowed one to write the reduced systems as physically interesting Hamiltonian systems. In the case of the KdV hierarchy, the q-representation for stationary flows has given rise to the Hénon–Heiles system [3, 4], the square eigenfunctions representation for restricted flows has furnished the Neumann and the Garnier systems [5, 6]. However the relation between dynamical systems which are obtained through different reduction techniques from the same soliton hierarchy is not clear; moreover a systematic way to find the second Hamiltonian formulation for stationary flows of any order, without the use of a Miura map, is still lacking.

The aim of this paper is to give a contribution in these directions. In particular:

(i) A bi-Hamiltonian formulation for stationary flows of the KdV hierarchy in a suitably extended phase space is derived in a systematic way. As an example, the bi-Hamiltonian structure of Hénon–Heiles-type systems is explicitly shown.
(ii) A map between stationary and restricted flows of the KdV hierarchy is obtained, based on the generating function of the Gelfand–Dickey (GD) polynomials. As an application, a map between an integrable Hénon–Heiles system and the Garnier system with two degrees of freedom is constructed.
(iii) An integrability criterion is proposed, which can be applied to both stationary and restricted flows. Though weaker than the bi-Hamiltonian formulation, it does not require the extension of the phase spaces.

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The paper is organized as follows. In section 2 we construct the stationary flows associated to the the KdV hierarchy through the kernel of the Poisson pencil. Using the generating function of GD polynomials as in [7], we give a bi-Lagrangian and a bi-Hamiltonian formulation of the Lax-Novikov stationary equations of any order; as an application, we exhibit a generalized Hénon–Heiles system.

In sections 3 and 4 we formulate the method of restricted flows in terms of the Poisson pencil instead of the spectral problem as in [5, 2]. This formulation allows us to explicitly construct a map between restricted and stationary flows, by means of an appropriate extension of the corresponding phase spaces. The previous map is specialized to the Hénon–Heiles and the Garnier systems.

In section 5 we show that the entire bi-Hamiltonian hierarchy of the Hénon–Heiles and the Garnier systems cannot be reduced from the extended to the standard phase space. For this reason, we propose an integrability criterion holding for a generic finite-dimensional Hamiltonian system. It generalizes the criterion introduced in [8] for the particular case of the Hénon–Heiles system. Though weaker than the bi-Hamiltonian scheme, it assures Liouville-integrability of a Hamiltonian system [9] in its standard phase space, i.e. without the introduction of supplementary coordinates. This criterion is applied to the generalized Hénon–Heiles system and to the Garnier system with two degrees of freedom.

Now we give some preliminaries, mainly to specify notation and terminology. Let M be a n-dimensional manifold. At any point $u \in M$, the tangent and cotangent spaces are denoted by T_uM and T_u^*M, the pairing between the two spaces by $\langle , \rangle : T_u^*M \times T_uM \to \mathbb{R}$. For each smooth function $f \in C^\infty(M)$, $\mathrm{d}f$ denotes the differential of f. M is said to be a Poisson manifold if it is endowed with a Poisson bracket $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, possibly a degenerate one; the associated Poisson tensor P is defined by $\{f, g\}(u) := \langle \mathrm{d}f(u), P_u\, \mathrm{d}g(u)\rangle$. So, at each point u, P_u is a linear map $P_u : T_u^*M \to T_uM$, skew-symmetric and with vanishing Schouten bracket [10]. A function $h \in C^\infty(M)$ with a non trivial differential $\mathrm{d}f \in Ker\, P$ is called a Casimir of P: $P_u\, \mathrm{d}f(u) = 0$. A map $\Phi : M \to M$ is a Poisson morphism if $\{f, g\} \circ \Phi = \{f \circ \Phi, g \circ \Phi\}$, for each $f, g \in C^\infty(M)$; Φ leaves invariant the Poisson tensor P: $P_{\Phi(u)} = \Phi_* P_u \Phi^*$, where Φ_* and Φ^* denote, respectively, the tangent and the cotangent maps associated to Φ. In particular, if the Poisson bracket is non-degenerate, i.e. if P is invertible, and the Poisson morphism is a diffeomorphism, Φ defines a symplectic (canonical) transformation. M is said to be a bi-Hamiltonian manifold if it is endowed with two Poisson tensors P_0 and P_1 such that the associated pencil $P^\lambda := P_1 - \lambda P_0$ be itself a Poisson tensor for any $\lambda \in \mathbb{C}$ [11, 12].

2. Stationary flows and Hénon–Heiles systems

2.1. KdV hierarchy and Gelfand–Dickey polynomials

Let M be a bi-Hamiltonian manifold: if the associated Poisson pencil $P^\lambda := P_1 - \lambda P_0$ admits as a Casimir a formal Laurent series $h(\lambda)$

$$h(\lambda) := \sum_{j \geqslant 0} h_j\, \lambda^{-j} \tag{2.1}$$

then h_0 is a Casimir of P_0 and the coefficients h_j $(j \geqslant 1)$ are the Hamiltonian functions of a hierarchy of bi-Hamiltonian vector fields X_j:

$$X_j = P_1 \mathrm{d}h_j = P_0 \mathrm{d}h_{j+1} \qquad (j \geqslant 0)\,. \tag{2.2}$$

At any point $u \in M$, the bi-Hamiltonian flows are given by $\mathrm{d}u/\mathrm{d}t_j = X_j(u)$, t_j being the evolution parameter of the jth flow. The vector fields (2.2) are Hamiltonian also with respect to the Poisson pencil P^λ. In fact the recursion relation (2.2) can be written as

$$X_j = P^\lambda \, \mathrm{d}h^{(j)}(\lambda) \qquad h^{(j)}(\lambda) := (\lambda^j h(\lambda))_+ \tag{2.3}$$

where the index + means the projection of a Laurent series onto the purely polynomial part.

Let M be the algebra of polynomials in $u, u_x, u_{xx}, \ldots$ ($u = u(x)$ is a C^∞ function of x and the subscript x means the derivative with respect to x), and let P_0 and P_1 be the two Poisson tensors of the KdV hierarchy [11]:

$$P_0 := \frac{\mathrm{d}}{\mathrm{d}x} \qquad P_1 := \frac{\mathrm{d}^3}{\mathrm{d}x^3} + 4u\frac{\mathrm{d}}{\mathrm{d}x} + 2u_x\,. \tag{2.4}$$

The gradients of the Casimirs of the associated Poisson pencil P^λ can be obtained searching for the 1-forms $v(\lambda) := \sum_{j\geqslant 0} v_j\, \lambda^{-j}$ which are solutions of the following equation:

$$B^\lambda(v(\lambda), v(\lambda)) = a(\lambda) \tag{2.5}$$

where $a(\lambda) = \sum_{j\geqslant -1} a_j \lambda^{-j}$, a_j are constant parameters and B^λ is the bilinear function

$$B^\lambda(w_1, w_2) := w_{1xx}w_2 + w_1 w_{2xx} - w_{1x}w_{2x} + (4u - \lambda)w_1 w_2\,. \tag{2.6}$$

In fact B^λ is related to the Poisson pencil through the relation

$$\frac{\mathrm{d}}{\mathrm{d}x} B^\lambda(w_1, w_2) = w_1\ P^\lambda w_2 + w_2\ P^\lambda w_1 \qquad (\forall\, w_1, w_2)\,. \tag{2.7}$$

Equation (2.5) can be solved developing the left-hand side as a Laurent series

$$B^\lambda(v(\lambda), v(\lambda)) = \sum_{k\geqslant -1} B_k \lambda^{-k} \tag{2.8}$$

so that, for each $a(\lambda)$, it furnishes the coefficients of the solution $v(\lambda)$ (unique up to a sign). The solution corresponding to $\overline{a}(\lambda) = -\lambda$ is the so-called basis solution $\overline{v}(\lambda)$; its first coefficients are

$$\overline{v}_0 = 1 \qquad \overline{v}_1 = 2u \qquad \overline{v}_2 = 2(u_{xx} + 3u^2) \qquad \overline{v}_3 = 2(u^{(4)} + 5u_x^2 + 10u_{xx}u + 10u^3) \tag{2.9}$$

and so on, namely the gradients of the first KdV Hamiltonians. In what follows we shall consider also the 1-form $v(\lambda) = c(\lambda)\overline{v}(\lambda)$, which is a solution of (2.5) for

$$a(\lambda) = -\lambda c^2(\lambda) \qquad c(\lambda) = 1 + \sum_{j\geqslant 1} c_j \lambda^{-j} \tag{2.10}$$

where the coefficients c_j are free parameters. In this case the first 1-forms of the hierarchy are $v_0 = 1$, $v_1 = \overline{v}_1 + c_1$, $v_2 = \overline{v}_2 + c_1\overline{v}_1 + c_2$, and so on.

The coefficient B_k in (2.8) can be expressed through the GD polynomials. For each Laurent series $v(\lambda)$ let us consider the functions $B^{(k)}(\lambda) := B^\lambda\left(v(\lambda), v^{(k)}(\lambda)\right)$, where $v^{(k)}(\lambda) := \left(\lambda^k v(\lambda)\right)_+$; these functions have the form

$$B^{(k)}(\lambda) = \lambda^{k+1} v_0^2 + \sum_{j=1}^{k-1} \lambda^{k-j}(p_{0j} - v_0 v_{j+1}) + \sum_{j\geqslant 0} \lambda^{-j} p_{jk} \qquad (j, k \in \mathbb{N}_0)\,. \tag{2.11}$$

It can be shown that

$$B_{-1} = -v_0^2 \qquad B_k = p_{0k} - v_0 v_{k+1} \quad (k \in \mathbb{N}_0)\,. \tag{2.12}$$

Furthermore, if $v(\lambda)$ is a solution of (2.5), the coefficients p_{jk} in (2.11) are polynomials in u and its x-derivatives. They will be referred to as Gelfand–Dickey (GD) polynomials and the function B^λ as their generating function.

The *fundamental* property of the GD polynomials, stemming from (2.11), (2.5), (2.7), (2.3), is the following relation with the gradients $v_j = \mathrm{d}h_j$ and the bi-Hamiltonian vector fields X_k:

$$\frac{\mathrm{d}}{\mathrm{d}x} p_{jk} = v_j X_k . \tag{2.13}$$

We report some GD polynomials to be used in what follows ($v_0 = 1$):

$$\begin{aligned} p_{00} &= 4u - v_1 \\ p_{01} &= 8uv_1 - v_1^2 - v_2 + 2v_{1xx} \\ p_{02} &= 4uv_1^2 + 8uv_2 - 2v_1v_2 - v_3 - v_{1x}^2 + 2v_1v_{1xx} + 2v_{2xx} \\ p_{12} &= 8uv_1v_2 - v_2^2 + 4uv_3 - v_1v_3 - v_4 + 2v_{1x}v_{2x} + 2v_2v_{1xx} + 2v_1v_{2xx} + v_{3xx} \\ p_{kk} &= 2v_{kxx}v_k - v_{kx}^2 + 4uv_k^2 . \end{aligned} \tag{2.14}$$

The GD polynomials corresponding to the basis solution $\overline{v}(\lambda)$ are the polynomials defined in [1, proposition 12.1.12].

2.2. *The method of stationary flows*

The method of *stationary flows* [13–15] was developed in order to reduce the flows of the KdV hierarchy onto the set M_n of fixed points of the nth flow X_n of the hierarchy:

$$M_n := \{u \mid X_n(u, u_x, \ldots, u^{(2n+1)}) = 0\} . \tag{2.15}$$

As M_n is odd-dimensional it cannot be a symplectic manifold; nevertheless we will show that it is a bi-Hamiltonian manifold: it will be referred to as *extended phase space*. Moreover, M_n is naturally foliated, on account of (2.2) and (2.4), by a one-parameter family of $2n$-dimensional submanifolds S_n given by

$$S_n := \{u \mid v_{n+1}(u, u_x, \ldots, u^{(2n)}) = c\} \tag{2.16}$$

(c being a constant parameter), which are invariant manifolds with respect to each vector field of the KdV hierarchy, due to the invariance of the 1-forms v_k. So M_n can be parametrized naturally by $v_1, \ldots, v_{n+1}$ and by their x-derivatives $v_{1x}, \ldots, v_{nx}$. We shall use these coordinates in what follows.

Here we perform two different stationary reductions of the KdV flows by improving the procedure introduced in [7]. On one side, we choose as a reduction submanifold $S_n^{(0)}$ just the leaf S_n of the foliation (2.16) corresponding to $c = 0$; it is a level set of the GD polynomial p_{0n}, due to (2.5), (2.8), (2.12). On account of equation (2.13), the GD polynomials p_{jn}, restricted to M_n, are also invariant with respect to each flow of the hierarchy; thus we can choose as a second reduction submanifold $S_n^{(1)}$ a level set of p_{nn}. The one-parameter family of the level sets of p_{nn} forms a foliation of the manifold M_n different from the previous one. Finally we construct the bi-Hamiltonian structure in the ground manifold M_n.

From the computational point of view, one proceeds as follows.

(i) Due to (2.3) and (2.5), the manifold M_n is defined by the solutions u of the equation

$$B^\lambda\big(v(\lambda), v^{(n)}(\lambda)\big) = \lambda^n a(\lambda) \tag{2.17}$$

where $v(\lambda) = \sum_{j=0}^n v_j\lambda^{-j}$, $a(\lambda) = \sum_{j=-1}^{2n} a_j\lambda^{-j}$. In particular, if $a(\lambda) = -\lambda c^2(\lambda)$, as in (2.10), M_n is given by

$$M_n = \left\{u \mid \bar{X}_n + \sum_{j=1}^{n} c_j \bar{X}_{n-j} = 0\right\} \tag{2.18}$$

i.e. by the solutions of the Lax-Novikov equations [13]. Taking into account (2.11) and choosing $a_{-1} = -1$, by equating in equation (2.17) the coefficients of λ^{n+1} we get $v_0^2 = 1$; from now on we put $v_0 = 1$. Moreover, equating the coefficients of the other powers of λ we get the following system:

$$p_{0k} - v_{k+1} = a_k \quad (k = 0, \ldots, n-1) \qquad p_{jn} = a_{n+j} \quad (j = 0, \ldots, n). \tag{2.19}$$

(ii) In order to obtain the first Poisson tensor P_0, we eliminate $u = v_1/2 + a_0/4$ from (2.19) using the first equation ($k = 0$) and we extract the system of n second-order ODEs in the v_j ($j = 1, \ldots, n$):

$$p_{0k} - v_{k+1} = a_k \quad (k = 1, \ldots, n-1) \qquad p_{0n} = a_n \tag{2.20}$$

which will be referred to as P_0-system. The remaining equations (2.19) will furnish a set of n independent integrals of motion. In order to obtain a second Poisson structure, we consider the following system: (P_1-system)

$$p_{0k} - v_{k+1} = a_k \quad (k = 1, \ldots, n-1) \qquad p_{nn} = a_{2n} \tag{2.21}$$

with u as above.

(iii) The system (2.20) can be written in Lagrangian form. For this purpose, we use the so-called Newton or r-representation introduced in [16]. Namely, we choose as new coordinates in $S_n^{(0)}$ the first n coefficients r_j of the formal series $r(\lambda) := \sqrt{v(\lambda)}$:

$$r_k = \Delta_{-k}\left(\sqrt{v(\lambda)}\right) \qquad (k = 1, \ldots, n) \tag{2.22}$$

where Δ_k means the coefficient of λ^k in a Laurent series. Taking into account (2.17), and observing that $2r_{n+1} = -\sum_{j=1}^{n} r_j r_{n+1-j}$, equations (2.20) are equivalent to

$$\left(\lambda^n\left(r_{xx} + \left(r_1 + \frac{a_0 - \lambda}{4}\right)r - \frac{a}{4r^3}\right)\right)_+ = 0. \tag{2.23}$$

This system is Lagrangian, with Lagrangian function

$$L_n^{(0)} = \Delta_{-(n+1)}\big(\mathcal{L}(\lambda; r(\lambda))\big) \tag{2.24}$$

where $\mathcal{L}\big(\lambda; w(\lambda)\big)$ is given, for each Laurent series $w(\lambda)$, by

$$\mathcal{L}(\lambda; w(\lambda)) := \frac{1}{2}\big(w_x(\lambda)\big)^2 - \frac{1}{2}\left(w_1 + \frac{a_0 - \lambda}{4}\right)w^2(\lambda) - \frac{a(\lambda)}{8w^2(\lambda)}. \tag{2.25}$$

The Lagrangian gradients

$$\frac{\delta}{\delta r_k} := \frac{\partial}{\partial r_k} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial r_{kx}}$$

of $L_n^{(0)}$ are

$$\frac{\delta L_n^{(0)}}{\delta r_k} = \Delta_{k-1}\left(\lambda^n\left(-r_{xx} - \left(r_1 + \frac{a_0 - \lambda}{4}\right)r + \frac{a}{4r^3}\right)\right)_+ \qquad (k = 1, \ldots, n). \tag{2.26}$$

We remark that it is also possible to put also the P_1-system (2.21) in Lagrangian form. For this purpose, we take as coordinates in $S_n^{(1)}$ $q_k = r_k$ ($k = 1, \ldots, n-1$) and $q_n = \sqrt{-v_n}$. By this choice the system (2.21) is equivalent to

$$\begin{aligned} &\frac{1}{2}q_n^2 + \left(\lambda^{n-1}\left(q_{xx} + \left(q_1 + \frac{a_0 - \lambda}{4}\right)q - \frac{a}{4q^3}\right)\right)_+ = 0 \\ &q_{nxx} + \left(q_1 + \frac{a_0}{4}\right)q_n - \frac{a_{2n}}{4q_n^3} = 0 \end{aligned} \tag{2.27}$$

where $\left(\lambda^{n-1}q(\lambda)\right)_+ := \left(\lambda^{n-1}\sqrt{v(\lambda)}\right)_+$. This is a Lagrangian system with Lagrangian

$$L_n^{(1)} = \Delta_{-n}\big(\mathcal{L}(\lambda; q(\lambda)\big) + \frac{1}{2}q_{nx}^2 - \frac{1}{2}(q_1 + \frac{a_0}{4})q_n^2 - \frac{a_{2n}}{8q_n^2}. \tag{2.28}$$

Indeed it can be verified that the Lagrangian gradients of $L_n^{(1)}$ are

$$\begin{aligned}
\frac{\delta L_n^{(1)}}{\delta q_1} &= \Delta_0\left(\lambda^{n-1}\left(-q_{xx} - \left(q_1 + \frac{a_0-\lambda}{4}\right)q + \frac{a}{4q^3}\right)\right)_+ - \frac{1}{2}q_n^2 \\
\frac{\delta L_n^{(1)}}{\delta q_k} &= \Delta_{k-1}\left(\lambda^{n-1}(-q_{xx} - \left(q_1 + \frac{a_0-\lambda}{4}\right)q + \frac{a}{4q^3}\right)\right)_+ \qquad (k = 2, \ldots, n-1) \\
\frac{\delta L_n^{(1)}}{\delta q_n} &= -q_{nxx} - \left(q_1 + \frac{a_0}{4}\right)q_n + \frac{a_{2n}}{4q_n^3}.
\end{aligned} \tag{2.29}$$

The two previous Lagrangian systems can be put in canonical Hamiltonian form. For the P_0-system the canonical momenta are $s_{n+1-k} = r_{kx}$ $(k = 1, \ldots, n)$ and the Hamiltonian function

$$H_n^{(0)} = \Delta_{-(n+1)}\left(\mathcal{H}(\lambda; r(\lambda), s(\lambda))\right) \tag{2.30}$$

where $s(\lambda) = \sum_{j=1}^n s_j\lambda^{-j}$ and $\mathcal{H}(\lambda; w(\lambda), z(\lambda))$ is given by

$$\mathcal{H}(\lambda; w(\lambda), z(\lambda)) = \frac{1}{2}z^2(\lambda) + \frac{1}{2}\left(w_1 + \frac{a_0-\lambda}{4}\right)w^2(\lambda) + \frac{a(\lambda)}{8w^2(\lambda)}. \tag{2.31}$$

For the P_1-system the canonical momenta are $p_n = q_{nx}$, $p_{n-k} = q_{kx}$ $(k = 1, \ldots, n-1)$, and the Hamiltonian function is

$$H_n^{(1)} = \Delta_{-n}\left(\mathcal{H}(\lambda; q(\lambda), p(\lambda))\right) + \frac{1}{2}p_n^2 + \frac{1}{2}\left(q_1 + \frac{a_0}{4}\right)q_n^2 + \frac{a_{2n}}{8q_n^2} \tag{2.32}$$

with $p(\lambda) = \sum_{j=1}^n p_j\lambda^{-j}$.

The two Hamiltonian functions depend, respectively, on the two sets of coordinates and momenta (r_k, s_k), (q_k, p_k) and on the two sets of free parameters $(a_0, \ldots, a_{n-1}, a_n)$ and $(a_0, \ldots, a_{n-1}, a_{2n})$.

(iv) Now let us consider the manifold M_n (2.18), which can be parametrized either by (r_k, s_k, a_n), or by (q_k, p_k, a_{2n}), with a_n and a_{2n} as additional dynamical variables in M_n. On this manifold one can trivially extend the canonical Poisson structures, the Hamiltonians and the vector fields associated with each one of the two systems as in [17]. In particular the vector fields can be extended in such a way that they are tangent to one of the foliations $S_{a_n}^{(0)}$ and $S_{a_{2n}}^{(1)}$. Taking into account, on one side, the relation between the two sets of coordinates through the original variables (v_k, v_{kx}), and on the other side the relation between the two integrals of motion a_n and a_{2n} through the GD polynomials p_{0n} and p_{nn}, a map $\Phi : M_n \to M_n, (r_k, s_k, a_n) \mapsto (q_k, p_k, a_{2n})$ can be systematically constructed. It relates the Hamiltonians and the vector fields of one system with the corresponding ones of the other system. Since this map is not a Poisson morphism, the extended canonical Poisson structures associated with one chart are mapped onto a Poisson structure different from the extended canonical structure associated with the other chart. If this second Poisson tensor is compatible with the extended canonical one, a bi-Hamiltonian formulation of the two systems is obtained.

In conclusion we can state the following:

Proposition 2.1. The P_0-system (2.20) and the P_1-system (2.21), written respectively in the coordinates r_k and q_k are natural Lagrangian systems. The corresponding canonical Hamiltonian systems

$$r_{kx} = \frac{\partial H_n^{(0)}}{\partial s_k} \qquad s_{kx} = -\frac{\partial H_n^{(0)}}{\partial r_k} \tag{2.33}$$

$$q_{kx} = \frac{\partial H_n^{(1)}}{\partial p_k} \qquad p_{kx} = -\frac{\partial H_n^{(1)}}{\partial q_k} \tag{2.34}$$

have n integrals of motion given by

$$K_j \equiv -\tfrac{1}{8} p_{jn_{|Y}} = a_{n+j} \quad (j = 1, \ldots, n) \qquad H_j \equiv -\tfrac{1}{8} p_{jn_{|X}} = a_{n+j} \quad (j = 0, \ldots, n-1)\,. \tag{2.35}$$

Moreover, the map $\Phi : M_n \to M_n$ in the extended phase space generates a second Poisson structure.

Remark 2.1. The symbols $|Y$ and $|X$ in (2.35) mean that, in the GD polynomials p_{jk}, the coordinates (v_k, v_{kx}) must be replaced by the canonical coordinates (r_k, s_k) and (q_k, p_k) respectively and that the first-order x-derivatives of momenta must be eliminated by means of the Hamiltonian dynamical equations (2.33), (2.34).

In the next subsection we shall give some applications of the results stated in this proposition.

2.3. *The bi-Hamiltonian structure of a Hénon–Heiles system*

We consider a generalized Hénon–Heiles system with two degrees of freedom.

Its Hamiltonian is

$$H_0 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + q_1^3 + \frac{1}{2} q_1 q_2^2 + \frac{a_4}{8q_2^2} + \frac{a_0}{2}\left(q_1^2 + \frac{1}{4} q_2^2\right) - \frac{a_1}{4} q_1 \tag{2.36}$$

where q_1, q_2, p_1, p_2 are the canonical coordinates and momenta and a_0, a_1, a_4 are free constant parameters. This Hamiltonian encompasses the two cases $a_0 = a_4 = 0$ and $a_0 = a_1 = 0$ introduced in [18]. Moreover H_0 is related with the Hamiltonian

$$H_H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left(A q_1'^2 + B q_2'^2\right) + q_1'^3 + \frac{1}{2} q_1' q_2'^2 + \frac{a_4}{8 q_2'^2} \tag{2.37}$$

through the map

$$q_1 = q_1' + \frac{A}{2} - 2B \qquad q_2 = q_2' \qquad a_0 = -2A + 12B \qquad a_1 = -A^2 + 16AB - 48B^2. \tag{2.38}$$

The function H_H is the Hamiltonian of a classical integrable Hénon–Heiles system [19] with the additional term $a_4/8q_2'^2$.

The function (2.36) is the Hamiltonian of the the vector field obtained reducing $X_0(u) = u_x$ to the stationary manifold M_2 given by the fixed points of the flow $X_2 + c_1 X_1 + c_2 X_0$

$$M_2 = \left\{u | u^{(5)} + 10 u_{xxx} u + 20 u_{xx} u_x + 30 u_x u^2 + c_1 (u_{xxx} + 6 u_x u) + c_2 u_x = 0\right\} \tag{2.39}$$

where $c_1 = -a_0/2$, $c_2 = -a_1/2 + a_0^2/4$.

It can be obtained specializing to the case $n = 2$ the Hamiltonian (2.32) of the P_1-system. In this case $H_2^{(1)} = H_0$ and the canonical coordinates and momenta are, respectively, $q_1 = v_1/2, q_2 = \sqrt{-v_2}$, $p_1 = q_{1x}$, $p_2 = q_{2x}$. The integrals of motion obtained by the reduction of the GD polynomials are

$$\begin{aligned} H_0 &\equiv -\tfrac{1}{8} p_{02_{|x}} \\ H_2 &\equiv -\tfrac{1}{8} p_{22_{|x}} = -\frac{a_4}{8} \\ H_1 &\equiv -\frac{1}{8} p_{12_{|x}} = p_2^2 q_1 - p_1 p_2 q_2 - \frac{1}{2} q_1^2 q_2^2 - \frac{1}{8} q_2^4 + \frac{a_4 q_1}{4 q_2^2} - \frac{a_0}{4} q_1 q_2^2 + \frac{a_1}{8} q_2^2 . \end{aligned} \tag{2.40}$$

The corresponding Hamiltonian vector fields will be denoted by $X_{j+1} := E\,\mathrm{d}H_j$ ($j = 0, 1, 2$); E being the canonical (4×4) Poisson matrix. The Hénon–Heiles vector field X_1 is:

$$X_1 = \left[p_1, p_2, -3q_1^2 - \frac{1}{2} q_2^2 - a_0 q_1 + \frac{a_1}{4}, -q_1 q_2 + \frac{a}{4 q_2^3} - \frac{a_0}{4} q_2 \right]^{\mathrm{T}} . \tag{2.41}$$

The second Hamiltonian formulation can be obtained specializing to the case $n = 2$ the Hamiltonian (2.30) of the P_0-system:

$$H_2^{(0)} = s_1 s_2 - \tfrac{5}{8} r_1^4 + \tfrac{5}{2} r_1^2 r_2 - \tfrac{1}{2} r_2^2 - \tfrac{1}{2} a_0 r_1^3 + \tfrac{3}{8} a_1 r_1^2 + a_0 r_1 r_2 - \tfrac{1}{4} a_2 r_1 - \tfrac{1}{4} a_1 r_2 \tag{2.42}$$

where the canonical coordinates (2.22) and momenta are, respectively, $r_1 = v_1/2$, $r_2 = v_2/2 - v_1^2/4$, $s_1 = r_{2x}$, $s_2 = r_{1x}$. The integrals of motion obtained by the reduction of the GD polynomials are

$$\begin{aligned} K_0 &\equiv -\tfrac{1}{8} p_{02_{|Y}} = -\tfrac{1}{8} a_2 \\ K_1 &\equiv -\tfrac{1}{8} p_{12_{|Y}} = H_2^{(0)} \\ K_2 &\equiv -\tfrac{1}{8} p_{22_{|Y}} = -s_2^2 r_2 + s_1 s_2 r_1 + \tfrac{1}{2} s_1^2 - \tfrac{1}{2} r_1^5 + 2 r_1 r_2^2 - \tfrac{3}{8} a_0 r_1^4 \\ &\quad + \tfrac{1}{4} a_1 r_1^3 - \tfrac{1}{2} a_0 r_1^2 r_2 + \tfrac{1}{2} a_1 r_1 r_2 + \tfrac{1}{2} a_0 r_2^2 - \tfrac{1}{8} a_2 r_1^2 - \tfrac{1}{4} a_2 r_2 \end{aligned} \tag{2.43}$$

and the corresponding Hamiltonian vector fields will be denoted by $Y_j := E\,\mathrm{d}K_j$.

Now we construct the bi-Hamiltonian structure of the Hénon–Heiles system. Let M_2 be the five-dimensional extended phase space parametrized by $(r_1, r_2, s_1, s_2; a_2)$ or $(q_1, q_2, p_1, p_2; a_4)$. It is convenient to make use of block notation. So, for example, we denote with $(r, s; a)$ the 5-tuple $(r_1, r_2, s_1, s_2; a_2)$, with $\tilde{X} = [\tilde{X}^r, \tilde{X}^s; \tilde{X}^a]^{\mathrm{T}}$ the generic vector field and with $\mathrm{d}\tilde{K} = [\partial \tilde{K}/\partial r, \partial \tilde{K}/\partial s; \partial \tilde{K}/\partial a]^{\mathrm{T}}$ the generic gradient of a function $\tilde{K}$ (the superscript T means transposition). In this notation a vector field $\tilde{X} = \tilde{P}\,\mathrm{d}\tilde{K}$ with Hamiltonian function $\tilde{K}$ with respect to a Poisson tensor $\tilde{P}$ will be written

$$\begin{bmatrix} \tilde{X}^r \\ \tilde{X}^s \\ \tilde{X}^a \end{bmatrix} = \begin{bmatrix} P^{rr} & P^{rs} & P^{ra} \\ P^{sr} & P^{ss} & P^{sa} \\ P^{ar} & P^{as} & P^{aa} \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{K}}{\partial r} \\ \frac{\partial \tilde{K}}{\partial s} \\ \frac{\partial \tilde{K}}{\partial a} \end{bmatrix} \tag{2.44}$$

where $P^{sr} = -(P^{rs})^{\mathrm{T}}, \ldots$, etc. From the definition of r_1, r_2 and q_1, q_2 in terms of v_1 and v_2, and from (2.40) and (2.43) one obtains the following map: $\Phi : M_2 \to M_2$, $(r, s; a_2) \mapsto (q, p; a_4)$

$$\begin{aligned} q_1 &= r_1 \qquad q_2 = (-2r_2 - r_1^2)^{1/2} \\ p_1 &= s_2 \qquad p_2 = -\frac{s_1 + r_1 s_2}{(-2r_2 - r_1^2)^{1/2}} \qquad a_4 = -8K_2 \end{aligned} \tag{2.45}$$

with K_2 given by equation (2.43). In these two charts let us consider the extended Hamiltonians $\tilde{H}_j$ and $\tilde{K}_j$, the vector fields $\tilde{X}_j$ ($\tilde{X}_j^r = X_j^r$, $\tilde{X}_j^s = X_j^s$, $\tilde{X}_j^a = 0$) and $\tilde{Y}_j$ ($\tilde{Y}_j^r = Y_j^r$, $\tilde{Y}_j^s = Y_j^s$, $\tilde{Y}_j^a = 0$), the extension of the canonical Poisson structure

$$\tilde{E} := \begin{pmatrix} \mathbf{0} & \mathbf{1} & 0 \\ -\mathbf{1} & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The following proposition holds:

Proposition 2.2. The action of the map $\Phi : M_2 \to M_2$ defined by (2.45) on the Hamiltonians $\tilde{H}_j$, the vector fields $\tilde{Y}_j$ and the Poisson tensor $\tilde{P}_0' := \tilde{E}$ is given by $\Phi^*(\tilde{H}_j) = \tilde{K}_j$, $\Phi_*(\tilde{Y}_j) = \tilde{X}_j$ and by

$$\tilde{P}_0 := \Phi_* \tilde{P}_0' \Phi^* = \begin{bmatrix} 0 & A & -8\tilde{X}_2^q \\ -A^{\mathrm{T}} & B & -8\tilde{X}_2^p \\ 8(\tilde{X}_2^q)^{\mathrm{T}} & 8(\tilde{X}_2^p)^{\mathrm{T}} & 0 \end{bmatrix} \tag{2.46}$$

where

$$A = \frac{1}{q_2^2}\begin{pmatrix} 0 & -q_2 \\ -q_2 & 2q_1 \end{pmatrix} \qquad B = \frac{1}{q_2^2}\begin{pmatrix} 0 & -p_2 \\ p_2 & 0 \end{pmatrix}.$$

Thus we have recovered in the extended phase space M_2 a second Poisson tensor $\tilde{P}_0$. We can check that $\tilde{P}_0$ is compatible with $\tilde{P}_1 = \tilde{E}$. Furthermore $\tilde{P}_0$ and $\tilde{P}_1$ give rise to the following bi-Hamiltonian hierarchy:

$$\tilde{X}_{j+1} := \tilde{P}_1 \, \mathrm{d}\tilde{H}_j = \tilde{P}_0 \, \mathrm{d}\tilde{H}_{j+1} \qquad (j = 0, 1) \tag{2.47}$$

the Hamiltonians $\tilde{H}_0$ and $\tilde{H}_2$ being Casimirs of $\tilde{P}_0$ and $\tilde{P}_1$, respectively.

3. Restricted flows and Garnier systems

The method of restricted flows was introduced in [20] as a *non-linearization* of the KdV spectral problem and was generalized in [5, 2]. We formulate this method putting the emphasis on the role of the GD polynomials and of their generating function; this formulation allows us to construct a map between stationary and restricted flows in the next section. In view of the applications, we begin by applying the method to the KdV hierarchy, recovering the Garnier system.

Let us consider the following system:

$$p_{00} - v_1 = a_0 \qquad P_0\left(v_1 - \sum_{j=1}^{n} \beta_j\right) = 0 \qquad P^{\lambda_k}\beta_k = 0 \quad (k = 1, \ldots, n) \tag{3.1}$$

where: $\lambda_1, \ldots, \lambda_n$ are distinct fixed parameters, $P^{\lambda_k} := P_1 - \lambda_k P_0$ (P_0 and P_1 being the two KdV Poisson tensors (2.4)). This is a system of $(n+2)$ equations in $u, v_1, \beta_1, \ldots, \beta_n$. The second equation will be referred to as the P_0*-restriction* of the first KdV flow $X_0 = P_0 v_1 = v_{1x}$, and the last n equations define the kernel of n Poisson tensors extracted from the Poisson pencil. On account of (2.14), (2.4) and(2.7) this system is equivalent to the following one:

$$u = \frac{v_1}{2} + \frac{a_0}{4} \qquad v_1 = \sum_{j=1}^{n} \beta_j + c \qquad B^{\lambda_k}(\beta_k, \beta_k) = f_k \tag{3.2}$$

where c and f_k are free parameters and B^λ is just the generating function (2.6) of the GD polynomials.

Using the first two equations to eliminate u and v_1 from the last n equations, one gets a system of n second-order ODEs for $\beta_1, \ldots, \beta_n$:

$$2\beta_{kxx}\beta_k - \beta_{kx}^2 + 2\beta_k^2\left(\sum_{j=1}^n \beta_j + d\right) - \lambda_k\beta_k^2 = f_k \qquad (k = 1, \ldots, n) \tag{3.3}$$

where $d := c + a_0/2$. Introducing the so-called eigenfunction variables $\psi_j = \sqrt{\beta_j}$ and the momenta $\chi_j = \psi_{jx}$, equations (3.3) can be written in canonical Hamiltonian form

$$\psi_{jx} = \frac{\partial \mathcal{K}_G}{\partial \chi_j} \qquad \chi_{jx} = -\frac{\partial \mathcal{K}_G}{\partial \psi_j} \qquad (j = 1, \ldots, n) \tag{3.4}$$

with Hamiltonian

$$\mathcal{K}_G = \frac{1}{2}\sum_{j=1}^n \chi_j^2 + \frac{1}{8}\left[\left(\sum_{k=1}^n \psi_j^2\right)^2 - \sum_{j=1}^n(\lambda_j - 2d)\psi_j^2 + \sum_{j=1}^n \frac{f_j}{\psi_j^2}\right]. \tag{3.5}$$

The corresponding Hamiltonian vector field $\mathcal{Y}_G = \mathcal{E}\,\mathrm{d}\mathcal{K}_G$ is

$$\mathcal{Y}_G = \left[\chi_j, -\frac{1}{2}(\psi_1^2 + \psi_2^2)\psi_j + \frac{1}{4}(\lambda_j - 2d)\psi_j + \frac{f_j}{4\psi_j^3}\right]^{\mathrm{T}} \qquad (j = 1, \ldots, n) \tag{3.6}$$

$\mathcal{E}$ being the $(2n \times 2n)$ canonical Poisson matrix. Equations (3.4) are just the equations of the Garnier system with n degrees of freedom [2]. A set of integrals of motion is

$$I_j = \chi_j^2 + \frac{\psi_j^2}{4}\left(2d - \lambda_j + \sum_{k=1}^n \psi_k^2\right) + \frac{f_j}{4\psi_j^2} + \sum_{\substack{k=1\\k\neq j}}^n \frac{1}{4\lambda_{jk}}\left(\frac{f_j\psi_k^2}{\psi_j^2} + \frac{f_k\psi_j^2}{\psi_k^2} + (\psi_j\chi_k - \psi_k\chi_j)^2\right) \tag{3.7}$$

with $\sum_{j=1}^n I_j = 2\mathcal{K}_G$. These integrals were obtained in [23] by means of a Lax representation; we shall recover them in the next section by the use of the generating function of the GD polynomials.

Let us consider the $(2n+1)$ extended phase space $\mathcal{M}_2$ with coordinates $(\psi_k, \chi_k; d)$ and the extended Hamiltonian $\tilde{\mathcal{K}}_G$, the vector field $\tilde{\mathcal{Y}}_G = \tilde{\mathcal{E}}\,\mathrm{d}\tilde{\mathcal{K}}_G$ with

$$\tilde{\mathcal{E}} = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n & 0 \\ -\mathbf{1}_n & \mathbf{0}_n & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this space the Garnier system has a second Hamiltonian structure given by

$$\tilde{\mathcal{P}}_1 := \begin{bmatrix} 0 & \Lambda - \psi \otimes \psi & 4\tilde{\mathcal{Y}}_G^{\psi} \\ -(\Lambda - \psi \otimes \psi)^{\mathrm{T}} & \chi \otimes \psi - \psi \otimes \chi & 4\tilde{\mathcal{Y}}_G^{\chi} \\ -4(\tilde{\mathcal{Y}}_G{}^{\psi})^{\mathrm{T}} & -4(\tilde{\mathcal{Y}}_G{}^{\chi})^{\mathrm{T}} & 0 \end{bmatrix} \tag{3.8}$$

where $\otimes$ denotes the tensor product, $\psi = [\psi_1, \ldots, \psi_n]^{\mathrm{T}}$, $\chi = [\chi_1, \ldots, \chi_n]^{\mathrm{T}}$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. This structure is an extension of the one constructed in [6] for $f_k = 0$, $(k = 1, \ldots, n)$. In view of the applications we specialize the above structure to the case $n = 2$, in the five-dimensional extended phase space $\mathcal{M}_2$ with coordinates $(\psi_1, \psi_2, \chi_1, \chi_2; d)$. The following proposition holds:

Proposition 3.1. The Garnier vector field $\tilde{\mathcal{Y}}_1 = \tilde{\mathcal{Y}}_G$ belongs to the following bi-Hamiltonian hierarchy:

$$\tilde{\mathcal{Y}}_{j+1} = \tilde{\mathcal{P}}_1\, \mathrm{d}\tilde{\mathcal{G}}_j = \tilde{\mathcal{P}}_0\, \mathrm{d}\tilde{\mathcal{G}}_{j+1} \qquad (j = 0, 1) \tag{3.9}$$

where the Hamiltonians $\tilde{\mathcal{G}}_j$ are given by

$$\begin{aligned} &\tilde{\mathcal{G}}_0 = \frac{d}{4} \qquad \tilde{\mathcal{G}}_1 = -(\lambda_1 + \lambda_2)\frac{d}{4} + \frac{1}{2}(\tilde{I}_1 + \tilde{I}_2) \\ &\tilde{\mathcal{G}}_2 = \lambda_1\lambda_2\frac{d}{4} - \frac{1}{2}(\lambda_1 + \lambda_2)(\tilde{I}_1 + \tilde{I}_2) + \frac{1}{2}(\lambda_1\tilde{I}_1 + \lambda_2\tilde{I}_2) \end{aligned} \tag{3.10}$$

$\tilde{\mathcal{G}}_0$ and $\tilde{\mathcal{G}}_2$ being Casimirs of $\tilde{\mathcal{P}}_0$ and $\tilde{\mathcal{P}}_1$, respectively, with $\tilde{I}_1, \tilde{I}_2$ being the extensions to $\mathcal{M}_2$ of the integrals of motion (3.7).

As in the case of the Hénon–Heiles system, a bi-Hamiltonian structure for the Garnier system seems to naturally exist only in its extended phase space. Nevertheless, in subsection 5.3 a realization of the integrability structure introduced in proposition 5.1 will be constructed in the original four-dimensional phase space.

4. A map between stationary and restricted flows

Now we shall construct a map between the nth stationary flow and the previous restricted flow of the KdV hierarchy. To this end we extend the corresponding phase spaces, regarding some free parameters in the Hamiltonian functions as additional dynamical variables.

4.1. The general case

As for the P_1-formulation of the stationary flow (2.34) we extend its phase space to a $(3n+1)$-dimensional space, $\tilde{M}_n$, with coordinates $(q_k, p_k; a_0, \ldots, a_{n-1}, a_{2n})$; analogously we consider the P_0-formulation of the first restricted flow (3.4) in the extended space $\tilde{\mathcal{M}}_n$ with coordinates $(\psi_k, \chi_k; f_1, \ldots, f_n, d)$.

Let us consider the solutions q_k of the dynamical equations (2.34); then $v^{(n)}(\lambda)$ given by

$$v^{(n)}(\lambda) = \lambda\left(q^2(\lambda)\right)^{(n-1)} - q_n^2 \tag{4.1}$$

with $q(\lambda) = 1 + \sum_{j=1}^{n} q_j\lambda^{-j}$, satisfies (2.17), and consequently satisfies the following equation:

$$B^{\lambda}\left(v^{(n)}(\lambda), v^{(n)}(\lambda)\right) = \lambda^{2n}a(\lambda) \tag{4.2}$$

where, as above, we put $u = v_1/2 + a_0/4$. So, for each n-tuple of distinct complex parameters λ_j, any solution $v^{(n)}(\lambda)$ of (4.2) fulfills the system

$$B^{\lambda_k}\left(v^{(n)}(\lambda_k), v^{(n)}(\lambda_k)\right) = \lambda_k^{2n}a(\lambda_k) \qquad (k = 1, \ldots, n) \tag{4.3}$$

where $v^{(n)}(\lambda_k) := v^{(n)}(\lambda)_{|\lambda=\lambda_k}$. In order to have a solution also satisfying the second equation (3.2), the Lagrange interpolation formula can be used [21, 22]. It allows us to represent the polynomial $v^{(n)}(\lambda)$ by

$$v^{(n)}(\lambda) = p(\lambda) + \sum_{j=1}^{n} \frac{p(\lambda)}{\lambda - \lambda_j}\beta_j \tag{4.4}$$

where $p(\lambda) = \prod_{j=1}^{n}(\lambda - \lambda_j)$, and

$$\beta_k = \frac{v^{(n)}(\lambda_k)}{p'(\lambda_k)} \qquad (k = 1, \ldots, n) \tag{4.5}$$

($p'(\lambda)$ means the derivative of $p(\lambda)$ with respect to λ).

Obviously the n functions β_k (4.5) are solutions of the following system:

$$2\beta_{kxx}\beta_k - \beta_{kx}^2 + 2\beta_k^2\left(\sum_{j=1}^{n}\beta_j + \frac{a_0}{2} - \sum_{j=1}^{n}\lambda_j\right) - \lambda_k\beta_k^2 = \frac{\lambda_k^{2n}a(\lambda_k)}{(p'(\lambda_k))^2} \qquad (k = 1, \ldots, n)\,. \tag{4.6}$$

Furthermore, β_k satisfy the so-called Bargmann constraint

$$\sum_{j=1}^{n}\left(\beta_j - \lambda_j\right) = v_1 \tag{4.7}$$

as one can verify by means of (4.4). Comparing (4.6) with (3.3), we can state the following:

Proposition 4.1. Let $\Psi : \tilde{M}_n \to \tilde{\mathcal{M}}_n, (q, p; a_0, \ldots, a_{n-1}, a_{2n}) \mapsto (\psi, \chi; f_1, \ldots, f_n, d)$ be the map:

$$\begin{aligned}
\psi_k &= \left(\frac{\sum_{j=0}^{n-1}\sum_{l=0}^{j} q_l q_{j-l}\,\lambda_k^{n-j} - q_n^2}{p'(\lambda_k)}\right)^{1/2} \\
\chi_k &= \frac{\sum_{j=1}^{n-1}\sum_{l=1}^{j} q_{j-l}p_{n-l}\,\lambda_k^{n-j} - q_n p_n}{\left(p'(\lambda_k)\left(\sum_{j=0}^{n-1}\sum_{l=0}^{j} q_l q_{j-l}\,\lambda_k^{n-j} - q_n^2\right)\right)^{1/2}} \\
f_k &= \frac{1}{(p'(\lambda_k))^2}\left(a_{2n} - 8\sum_{j=0}^{n} H_{n-j}\,\lambda_k^j + \sum_{j=n+1}^{2n+1} a_{2n-j}\,\lambda_k^j\right) \\
d &= \frac{a_0}{2} - \sum_{j=1}^{n}\lambda_j \\
&\qquad (k = 1, \ldots, n)
\end{aligned} \tag{4.8}$$

where H_j are the Hamiltonian functions (2.35). If (q_k, p_k) are solutions of the stationary flows (2.34), then (ψ_k, χ_k) are solutions of the Garnier system (3.4) for f_k and d given by (4.8).

Remark 4.1. The function B^λ is also a generating function of integrals of motion for the Garnier system. Indeed evaluating the function B^λ by means of (4.4) and eliminating the first x-derivatives of χ_k by means of the Hamilton equations (3.4), one gets

$$4\sum_{j=1}^{n}\frac{I_j}{\lambda - \lambda_j} + \sum_{j=1}^{n}\frac{f_j}{(\lambda - \lambda_j)^2} + 2d - \lambda = \frac{\lambda^{2n}\hat{a}(\lambda)}{(p(\lambda))^2} \tag{4.9}$$

where I_j are the functions (3.7). Taking in this equation the residues at $\lambda = \lambda_j$ it follows that the functions I_j are integrals of motion along the flow (3.4).

4.2. The map between the Hénon–Heiles and the Garnier system

Now we specialize the map of proposition 4.1 to the Hénon–Heiles and the Garnier systems with two degrees of freedom: we obtain the surprising result that the Hénon–Heiles vector field is mapped onto the Garnier vector field. Let us consider the seven-dimensional phase space of the Hénon–Heiles system $\tilde{M}_2$ with coordinates $(q, p; a_0, a_1, a_4)$. Similarly, for the Garnier systems let us select the parameters f_1, f_2, d and enlarge the phase space to a seven-dimensional phase space $\tilde{\mathcal{M}}_2$, with coordinates $(\psi, \chi; f_1, f_2, d)$. It is easy to prove the following:

Proposition 4.2. Let $\Psi : \tilde{M}_2 \to \tilde{\mathcal{M}}_2, (q, p; a_0, a_1, a_4) \mapsto (\psi, \chi; f_1, f_2, d)$ be defined by

$$\begin{aligned}
&\psi_1 = \lambda_{12}^{-1/2}(\lambda_1^2 + 2\lambda_1 q_1 - q_2^2)^{1/2} \qquad \psi_2 = \lambda_{12}^{-1/2}(-\lambda_2^2 - 2\lambda_2 q_1 + q_2^2)^{1/2} \\
&\chi_1 = \frac{(\lambda_1 p_1 - q_2 p_2)}{\left(\lambda_{12}\,(\lambda_1^2 + 2\lambda_1 q_1 - q_2^2)\right)^{1/2}} \qquad \chi_2 = \frac{(\lambda_2 p_1 - q_2 p_2)}{\left(\lambda_{12}\,(-\lambda_2^2 - 2\lambda_2 q_1 + q_2^2)\right)^{1/2}} \\
&f_1 = \lambda_{12}^{-2}\left(-\lambda_1^5 + a_0\lambda_1^4 + a_1\lambda_1^3 - 8H_0\lambda_1^2 - 8H_1\lambda_1 + a_4\right) \\
&f_2 = \lambda_{12}^{-2}\left(-\lambda_2^5 + a_0\lambda_2^4 + a_1\lambda_2^3 - 8H_0\lambda_2^2 - 8H_1\lambda_2 + a_4\right) \qquad d = \frac{a_0}{2} - (\lambda_1 + \lambda_2)
\end{aligned} \tag{4.10}$$

where $\lambda_{12} = \lambda_1 - \lambda_2$. The tangent map Ψ_* maps the extended Hénon–Heiles vector fields $\tilde{X}_1, \tilde{X}_2$ (2.47) onto the extended Garnier vector fields $\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2$ (3.10):

$$\Psi_*\left(\tilde{X}_1\right) = \tilde{\mathcal{Y}}_1 \qquad \Psi_*\left(\tilde{X}_2\right) = \tilde{\mathcal{Y}}_2\,. \tag{4.11}$$

Moreover the pull-back of the Garnier integrals of motion $\mathcal{G}_1$ and $\mathcal{G}_2$ are integrals of motion for the Hénon–Heiles system

$$\begin{aligned}
\Psi^*(\mathcal{G}_1) &= -\frac{1}{8}(\lambda_1^2 + \lambda_2^2) + \frac{a_0}{8}(\lambda_1 + \lambda_2) + \frac{a_1}{8} \\
\Psi^*(\mathcal{G}_2) &= \lambda_{12}^{-2}\,(2\lambda_1\lambda_2 H_0 + (\lambda_1 + \lambda_2)H_1 + 2H_2) \\
&\quad + \frac{\lambda_{12}^{-2}\lambda_1\lambda_2}{4}\left((\lambda_1^3 + \lambda_2^3) - \frac{a_0}{2}(\lambda_1^2 + \lambda_2^2) - \frac{a_1}{2}(\lambda_1 + \lambda_2)\right).
\end{aligned} \tag{4.12}$$

The action of the map Ψ on the Poisson tensor $\tilde{E}$ of the Hénon–Heiles system, furnishes a new Poisson tensor for the Garnier system compatible with $\tilde{\mathcal{E}}$. Moreover the action of Ψ on the Poisson tensor $\tilde{P}_0$ is given by

$$\Psi^*\tilde{P}_0\Psi_* = \lambda_{12}^{-2}\begin{bmatrix} 0 & \mathcal{A} & 0 \\ -\mathcal{A}^{\mathrm{T}} & \mathcal{B} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.13}$$

where

$$\begin{aligned}
\mathcal{A} &= \frac{1}{\psi_1^2\psi_2^2}\begin{bmatrix} \psi_2^2(\psi_1^2 + \psi_2^2 + \lambda_1 - \lambda_2) & -\psi_1\psi_2(\psi_1^2 + \psi_2^2) \\ -\psi_1\psi_2(\psi_1^2 + \psi_2^2) & \psi_1^2(\psi_1^2 + \psi_2^2 + \lambda_2 - \lambda_1) \end{bmatrix} \\
\mathcal{B} &= \frac{\psi_1^2+\psi_2^2}{\psi_1^2\psi_2^2}\begin{bmatrix} 0 & (\chi_2\psi_1 - \chi_1\psi_2) \\ -(\chi_2\psi_1 - \chi_1\psi_2) & 0 \end{bmatrix}.
\end{aligned} \tag{4.14}$$

So the map Ψ is not a Poisson morphism. However, according to (4.11), the orbits of the Hénon–Heiles system are mapped onto the orbits of the Garnier system.

5. A new integrability structure

5.1. The reduced structures of the Hénon–Heiles and the Garnier systems

In order to have a bi-Hamiltonian hierarchy also in the original phase space for the Hénon–Heiles and the Garnier systems, one can try to apply the reduction techniques known from the literature [10, 24]. In particular, two methods can be followed: a *restriction* to the standard phase space or a *projection* onto it. However, in both cases, these attempts fail.

As for the Hénon–Heiles system, if the restriction submanifold is chosen to be a leaf $S^{(1)}_{a_4}$ of the second natural foliation in M_2, the Hamiltonians $\tilde{H}_j$, the vector fields $\tilde{X}_j$ and the Poisson structure $\tilde{P}_1$ can be trivially restricted respectively to H_j, X_j and E; but it turns out that $\tilde{P}_0$ cannot be restricted. So two integrable Hamiltonian vector fields are obtained in $S^{(1)}_{a_4}$ but not a bi-Hamiltonian hierarchy.

If $\Pi : M_2 \to S_2, (q_1, q_2, p_1, p_2; a_{2n}) \mapsto (q_1, q_2, p_1, p_2)$ is the projection map, the Hamiltonians $\tilde{H}_j$ and the vector fields $\tilde{X}_j$ cannot be projected onto S_2, because they depend on the fibre coordinate. Instead, the Poisson tensors $\tilde{P}_0$ and $\tilde{P}_1$ are projected onto

$$P_H := \Pi_* \tilde{P}_0 \Pi^* = \begin{bmatrix} 0 & A \\ -A^{\mathrm{T}} & B \end{bmatrix} \qquad \Pi_* \tilde{P}_1 \Pi^* = E \tag{5.1}$$

with A, B as in proposition 2.2. Because these operators are compatible and invertible, one obtains the following Nijenhuis tensor [25]:

$$N_H := P_H E^{-1} = \begin{bmatrix} A & 0 \\ B & A^T \end{bmatrix} \tag{5.2}$$

and consequently the hierarchy of Poisson tensors $P_k := N_H^k P_H$, $k \in \mathbb{Z}$. However, these tensors are not invariant along the flow of the Hénon–Heiles vector field X_1, equation (2.41). In other words X_1 is neither a symmetry of P_0 nor of P_1, so that these tensors cannot generate a bi-Hamiltonian hierarchy starting from X_1.

As in the case of the Hénon–Heiles system, one cannot reduce the bi-Hamiltonian structure of the Garnier system with n degrees of freedom onto the restricted phase space. If $\Pi : \mathcal{M}_n \to S^{(1)}_n, (\psi_k, \chi_k; d) \mapsto (\psi_k, \chi_k)$ is the projection map, the Poisson tensor $\tilde{\mathcal{P}}_0$ and $\tilde{\mathcal{P}}_1$ are projected onto two compatible tensors

$$\Pi_* \tilde{\mathcal{P}}_0 \Pi^* = \mathcal{E} \qquad \mathcal{P}_G := \Pi_* \tilde{\mathcal{P}}_1 \Pi^* = \begin{bmatrix} 0 & \Lambda - \psi \otimes \psi \\ -(\Lambda - \psi \otimes \psi)^{\mathrm{T}} & \chi \otimes \psi - \psi \otimes \chi \end{bmatrix}. \tag{5.3}$$

They give rise to the Nijenhuis tensor $\mathcal{N}_G := \mathcal{P}_G \mathcal{E}^{-1}$ together with the hierarchy of Poisson tensor fields $\mathcal{P}_k := \mathcal{N}_G^k \mathcal{E}$, $k \in \mathbb{Z}$. However, these tensor fields are not invariant along the flow of the Garnier vector field $\mathcal{Y}_G$ (3.6), so they do not generate a bi-Hamiltonian hierarchy starting from $\mathcal{Y}_G$.

5.2. A new integrability criterion

In the previous subsection we have put into evidence some problems arising in the geometrical reduction of a bi-Hamiltonian structure from an extended phase space onto the original one. As an alternative construction, here we introduce a new integrability scheme, weaker than the bi-Hamiltonian one, but living in the standard phase space. We shall define this new structure for a generic Hamiltonian system with n degrees of freedom; for $n = 2$ it coincides with the one introduced in [8] for the Hénon–Heiles system with the Hamiltonian (2.37) and $a_4 = 0$. As new examples of this integrability structure, the case of the Garnier system with two degrees of freedom will be discussed here whereas multidimensional extensions of the Hénon–Heiles system will be presented elsewhere.

Proposition 5.1. Let M be a $2n$-dimensional Poisson manifold equipped with a Poisson tensor Q_0, and Z_0 a Hamiltonian vector field with Hamiltonian h_0: $Z_0 = Q_0 \, \mathrm{d}h_0$. Let there exist a tensor $\mathcal{N} : TM \to TM$ and a skew-symmetric tensor $Q_1 : T^*M \to TM$ such that

$$Q_1 = \mathcal{N} Q_0 \,. \tag{5.4}$$

Denote by Z_i and α_i the vector fields and the 1-forms obtained, respectively, by the iterated action of the tensor $\mathcal{N}$ on Z_0 and its adjoint $\mathcal{N}^* : T^*M \to T^*M$ on $\alpha_0 := \mathrm{d}h_0$

$$Z_i := \mathcal{N}^i Z_0 \qquad \alpha_i := \mathcal{N}^{*i} \alpha_0 \qquad (i = 1, \ldots, n-1) \,. \tag{5.5}$$

Let there exist $n-1$ independent functions h_i $(i = 1, \ldots, n-1)$ and $(n^2 + n - 2)/2$ functions μ_{ij} $(i = 1, \ldots, n-1; 0 \leqslant j \leqslant i)$ with $\mu_{00} = 1$, $\mu_{ii} \neq 0$ $(i = 1, \ldots, n-1)$, such that the 1-forms α_i can be written as

$$\alpha_i = \sum_{j=0}^{i} \mu_{ij} \, \mathrm{d}h_j \qquad (i = 1, \ldots, n-1) \,. \tag{5.6}$$

Under the previous assumptions the following results hold:
(i) the vector fields Z_i satisfy the recursion relations

$$Z_{i+1} = Q_0 \alpha_{i+1} = Q_1 \alpha_i \qquad (i = 0, \ldots, n-2). \tag{5.7}$$

(ii) the functions h_i are in involution with respect to the Poisson bracket defined by Q_0 and they are constants of motion for the fields Z_k

$$\{h_i, h_j\}_{Q_0} = 0 \qquad \mathcal{L}_{Z_k}(h_i) = 0 \tag{5.8}$$

where $\mathcal{L}_{Z_k}$ denotes the Lie derivative with respect to the vector field Z_k.
(iii) the Hamiltonian system corresponding to the vector field Z_0 is Liouville-integrable. In addition if Q_1 is a Poisson tensor field, then also Z_1 is an integrable Hamiltonian vector field and the functions h_i are in involution also with respect to the Poisson bracket defined by Q_1.

Proof. (i) From (5.4) and the skew-symmetry of Q_0 and Q_1 it follows that $Q_0 \mathcal{N}^* = \mathcal{N} Q_0$ and $Q_1 \mathcal{N}^* = \mathcal{N} Q_1$. Then

$$Z_1 - Q_0 \alpha_1 = Z_1 - Q_0 \mathcal{N}^* \alpha_0 = Z_1 - \mathcal{N} Q_0 \alpha_0 = 0 \tag{5.9}$$

and the first relation (5.7) is proved by induction since it is

$$Z_{i+1} - Q_0 \alpha_{i+1} = \mathcal{N} Z_i - Q_0 \mathcal{N}^* \alpha_i = \mathcal{N}(Z_i - Q_0 \alpha_i) \,. \tag{5.10}$$

The second relation (5.7) follows from

$$Z_{i+1} - Q_1 \alpha_i = \mathcal{N} Z_i - Q_1 \alpha_i = \mathcal{N}(Z_i - Q_0 \alpha_i) \,. \tag{5.11}$$

(ii) By (5.6), the gradients $\mathrm{d}h_k$ can be expressed for any k in terms of $\mathrm{d}h_0$

$$\mathrm{d}h_k = \left(\sum_{i=0}^{k} \nu_{ki} \mathcal{N}^{*^i}\right) \mathrm{d}h_0 \tag{5.12}$$

where ν_{ki} are the elements of the matrix a^{-1}, a being the lower-triangular matrix defined by $a_{ij} = \mu_{ij}$ $(i \geqslant j)$, $a_{ij} = 0\,(i < j)$, $(i, j = 0, \ldots, n-1)$. Thus

$$\begin{aligned}\{h_i, h_j\}_{Q_0} : &= \langle \mathrm{d}h_i, Q_0 \mathrm{d}h_j \rangle \\ &= \sum_{a=0}^{i} \sum_{b=0}^{j} \nu_{ia} \nu_{jb} \langle \mathcal{N}^{*^a} \mathrm{d}h_0, Q_0 \mathcal{N}^{*^b} \mathrm{d}h_0 \rangle \\ &= \sum_{a=0}^{i} \sum_{b=0}^{j} \nu_{ia} \nu_{jb} \langle \mathrm{d}h_0, \mathcal{N}^{a+b} Q_0 \mathrm{d}h_0 \rangle \end{aligned} \tag{5.13}$$

and the first relation (5.8) follows from the skew-symmetry of the tensor $\mathcal{N}^m Q_0$ for any m. Furthermore

$$\begin{aligned}\mathcal{L}_{Z_k}(h_i) &= \langle \mathrm{d}h_i, Q_0 \alpha_{k-1} \rangle \\ &= \left\langle \mathrm{d}h_i, Q_0 \sum_{j=0}^{k} \mu_{kj} \mathrm{d}h_j \right\rangle \\ &= \sum_{j=0}^{k} \mu_{kj} \{h_i, h_j\}_{Q_0} \\ &= 0. \end{aligned} \tag{5.14}$$

(iii) Since Z_0 is a Hamiltonian vector field, it is Liouville-integrable on account of the previous result. Moreover, since it is

$$\begin{aligned}\{h_i, h_j\}_{Q_1} : &= \langle \mathrm{d}h_i, Q_1 \mathrm{d}h_j \rangle \\ &= \sum_{a=0}^{i} \sum_{b=0}^{j} \nu_{ia} \nu_{jb} \langle \mathcal{N}^{*^a} \mathrm{d}h_0, Q_1 \mathcal{N}^{*^b} \mathrm{d}h_0 \rangle \\ &= \sum_{a=0}^{i} \sum_{b=0}^{j} \nu_{ia} \nu_{jb} \langle \mathrm{d}h_0, \mathcal{N}^{a+b} Q_1 \mathrm{d}h_0 \rangle \\ &= 0 \end{aligned} \tag{5.15}$$

it follows that if Q_1 is also a Poisson tensor, $\{,\}_{Q_1}$ is a Poisson bracket, Z_1 is a Hamiltonian vector field and then it is Liouville-integrable. □

Remark 5.1. The recursion scheme and the integrability of the vector field Z_0 do not require that the skew-symmetric tensor Q_1 be a Poisson tensor; so M is a Poisson manifold, not a bi-Hamiltonian one.

In view of the applications of the next subsection, it may be worthwhile to remark that the results of proposition 5.1 hold true if the role of Q_0 and Q_1 are interchanged; to be more precise, one can prove (just as for proposition 5.1):

Proposition 5.2. The integrability scheme of proposition 5.1 is still valid if Q_0 is skew-symmetric, Q_1 is a Poisson tensor and the role of Z_0 is now played by $Z_1 = Q_1 \mathrm{d}h_0$. The involution relations (5.8) become $\{h_i, h_j\}_{Q_1} = 0$.

5.3. The integrability structure of the Hénon–Heiles and the Garnier systems

In subsection 5.1 we recovered by projection onto the quotient manifold S_2 the Nijenhuis tensor (5.2) and a hierarchy of compatible Poisson tensors; however, it is not possible to associate to these tensors and to the Hénon–Heiles vector field X_1 (2.41) a bi-Hamiltonian hierarchy of vector fields. Nevertheless it is possible to use these elements to construct an example of the integrability structure introduced in proposition 5.2. For this purpose, let us make the following choices:

(i) $Q_1 = E$, the vector field $Z_1 := X_1$ (2.41) with Hamiltonian $h_0 := H_0$ (2.36);
(ii) the tensor field $\mathcal{N} := N_H$ (5.2) and $Q_0 := P_{-2} = N_H^{-2} P_H$, with P_H as in (5.1);
(iii) the function $h_1 := H_1$ (2.40) and the functions μ_{ij} as $\mu_{10} = 0, \mu_{11} = 1/q_2^2$;

then it is immediate to check that the conditions of proposition 5.2 are satisfied. Moreover the vector field $Z_0 := Q_0 \,\mathrm{d}h_0 = P_{-2}\,\mathrm{d}H_0$ is a new integrable vector field:

$$Z_0 = \begin{bmatrix} -2p_1q_1 - p_2q_2 \\ -p_1q_2 \\ -p_2^2 + 6q_1^3 + 2q_1q_2 - \frac{a_4}{4q_2^2} - \frac{a_1}{2}q_1 - 2a_0q_1^2 + \frac{a_0}{4}q_2^2 \\ p_1p_2 + \frac{q_2^3}{2} + 3q_1^2q_2 - \frac{a_1}{4}q_2 + a_0q_1q_2 \end{bmatrix}. \tag{5.16}$$

This integrability structure is related, through the map (2.38), to the one introduced in [8] for the Hamiltonian (2.37) with $a_4 = 0$.

For the Garnier system with two degrees of freedom one can construct an example of the integrability structures of proposition 5.1. Indeed if one uses the elements of subsection 5.1 and makes the following choices:

(i) $Q_0 := \mathcal{E}$, $h_0 := \tilde{\mathcal{G}}_1$ (3.11) , $Z_0 := \mathcal{Y}_G$ (3.6) ;
(ii) $\mathcal{N} := \mathcal{N}_G^{-1} = \mathcal{E}\,\mathcal{P}_G^{-1}$, with $\mathcal{P}_G$ as in (5.3) , $\mathcal{Q}_1 := \mathcal{P}_{-1} = \mathcal{N}_G^{-1}\,\mathcal{E}$;
(iii) the functions $h_1 := \tilde{\mathcal{G}}_2$ (3.11) , $\mu_{10} = 0$, $\mu_{11} = -\frac{\lambda_1^2\lambda_2^2}{\lambda_2\psi_1^2 + \lambda_1\psi_2^2 - \lambda_1\lambda_2}$;

then the conditions of proposition 5.1 are satisfied. Moreover the vector field $Z_1 := \mu_{11}\mathcal{Y}_2$ is a new integrable vector field ($\mathcal{Y}_2$ is the restriction to the submanifold of $\mathcal{M}_2$, d = cost, of the vector field $\tilde{\mathcal{Y}}_2$ (3.10)).

At last, we compute how the map between the standard phase spaces of the Hénon–Heiles and of the Garnier systems, induced by the map (4.10), acts on the recursion operators of the previous integrability structures.

Proposition 5.3. Let us consider the map $\Psi : (q_1, q_2, p_1, p_2) \mapsto (\psi_1, \psi_2, \chi_1, \chi_2)$

$$\psi_1 = \lambda_{12}^{-1/2}(\lambda_1^2 + 2\lambda_1q_1 - q_2^2)^{1/2} \qquad \psi_2 = (\lambda_{12})^{-1/2}(-\lambda_2^2 - 2\lambda_2q_1 + q_2^2)^{1/2}$$
$$\chi_1 = \frac{(\lambda_1p_1 - q_2p_2)}{\left(\lambda_{12}\,(\lambda_1^2 + 2\lambda_1q_1 - q_2^2)\right)^{1/2}} \qquad \chi_2 = \frac{(\lambda_2p_1 - q_2p_2)}{\left(\lambda_{12}\,(-\lambda_2^2 - 2\lambda_2q_1 + q_2^2)\right)^{1/2}}. \tag{5.17}$$

The map Ψ relates the recursion operators of the Hénon–Heiles and of the Garnier systems: $\Psi_* N_H = \mathcal{N}_G^{-1}\Psi_*$.

6. Concluding remarks

In this paper we have derived a bi-Hamiltonian formulation for stationary flows, and for the first restricted flows of the KdV hierarchy. Our approach amounts to respectively searching the kernel of the Poisson pencil and n-Poisson structures extracted from the Poisson pencil of the KdV hierarchy. In this approach the generating function of the GD polynomials plays a relevant role. Moreover it allows us to construct a map between stationary flows and restricted flows; in the case of the fifth-order stationary KdV equation, this map relates solutions of the Hénon–Heiles system to solutions of the Garnier system. However, to obtain these results one must extend the phase space of the reduced flows by means of some free parameters naturally contained in the corresponding Hamiltonian functions. This difficulty can be overcome, at least if one analyses the complete integrability of a Hamiltonian system without requiring an explicit knowledge of a bi-Hamiltonian structure. For this purpose, we have introduced a new integrability scheme in the standard phase space, which implies Liouville integrability of the reduced Hamiltonian systems. For brevity we have applied this scheme only to the Hénon–Heiles and the Garnier systems with two degrees of freedom. Other examples such as Hénon–Heiles type systems with three and four degrees of freedom, constructed by means of the reduction method of section 2, will be discussed elsewhere.

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